Exercise 2.3.1. Let $x_n \geq 0$ for all $n \in \mathbf{N}$.

(a) If $(x_n) \to 0$, show that $(\sqrt{x_n}) \to 0$.

(b) If $(x_n) \to x$, show that $(\sqrt{x_n}) \to \sqrt{x}$.

Scratchwork – Know $x_n \to 0$, $x_n \geq 0$
$\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ s.t. $\forall n \geq N$,
$|x_n| = |x_n - 0| < \varepsilon$ given

Want: $\forall \varepsilon > 0 \quad |\sqrt{x_n} - 0| < \varepsilon$

$$\iff |\sqrt{x_n}| < \varepsilon$$

$$\iff \sqrt{x_n} < \varepsilon$$

$$\iff \underbrace{x_n}_{= |x_n|} < \varepsilon^2$$

Can get to this from given
(Just use ε^2 as my starting positive # in given defn.)

(a) Proof: $\forall \varepsilon > 0$, Since $x_n \to 0$, $\exists N \in \mathbb{N}$
s.t. $\forall n \geq N$, $|x_n| = |x_n - 0| < \varepsilon^2$.
Then $\sqrt{x_n} = |\sqrt{x_n}| = |\sqrt{x_n} - 0| < \varepsilon$.

Thus $\lim \sqrt{x_n} = 0$. $\boxed{\checkmark}$

(b) Now: $x_n \to x > 0$, $x_n \geq 0\ \forall n$

Scratch

Want $\lim \sqrt{x_n} = \sqrt{x}$

$\iff \forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n \geq N$,

$|\sqrt{x_n} - \sqrt{x}| < \varepsilon$.

Given: $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n \geq N$,

$|x_n - x| < \varepsilon$.

Want: $|\underbrace{\sqrt{x_n}}_{\geq 0} - \underbrace{\sqrt{x}}_{\geq 0}| < \varepsilon$.

$\sqrt{x_n - x} \stackrel{?}{=} \ldots$? (If $\sqrt{x_n x} = \sqrt{x_n}\sqrt{x}$

$A^2 - B^2 = (A-B)(A+B)$.

multiply $|(x_n - x_n)| = |(\sqrt{x_n} - \sqrt{x})(\sqrt{x_n} + \sqrt{x})|$

$= |\sqrt{x_n} - \sqrt{x}|\,|\sqrt{x_n} + \sqrt{x}|$

Mult both sides by $\sqrt{x_n} + \sqrt{x} = |\sqrt{x_n} + \sqrt{x}|$

$\Rightarrow |\sqrt{x_n} - \sqrt{x}|\,|\sqrt{x_n} + \sqrt{x}| < \varepsilon|\sqrt{x_n} + \sqrt{x}|$

$\Rightarrow |x_n - x| < \underbrace{\varepsilon(\sqrt{x_n} + \sqrt{x})}$,

$< \varepsilon(\sqrt{x+1} + \sqrt{x})$

Would work if x_n's are chosen so they are within 1 of x

$\varepsilon(\sqrt{x})$

$$\underbrace{\underbrace{|x_n - x| < \varepsilon\sqrt{x}}_{\text{if I have this}} < \varepsilon(\sqrt{x_n}+\sqrt{x})}_{\text{Then I get this.}}$$

We can use given, but with ε replaced by $\varepsilon\sqrt{x} > 0$.

(b) **Proof:** Given any $\varepsilon > 0$, because $x_n \to x$ with $x_n \geq 0$ and $x > 0$, $\exists N \in \mathbb{N}$ s.t. if $n \geq N$, then

$$|x_n - x| < \varepsilon\sqrt{x}.$$

Then $$|x_n - x| < \varepsilon(\sqrt{x_n}+\sqrt{x})$$

$$\Rightarrow |(x_n - x)| = |(\sqrt{x_n}-\sqrt{x})(\sqrt{x_n}+\sqrt{x})| < \varepsilon(\sqrt{x_n}+\sqrt{x})$$

$$\Rightarrow \quad |\sqrt{x_n}-\sqrt{x}|\;|\sqrt{x_n}+\sqrt{x}| < \varepsilon(\sqrt{x_n}+\sqrt{x})$$

$$\Rightarrow |\sqrt{x_n}-\sqrt{x}|\,(\underset{\to 0}{\sqrt{x_n}+\sqrt{x}}) < \varepsilon\,(\underset{\to 0}{\sqrt{x_n}+\sqrt{x}})$$

$$\Rightarrow \quad |\sqrt{x_n}-\sqrt{x}| < \varepsilon.$$

Thus, $\lim \sqrt{x_n} = \sqrt{x}$. $\boxed{\checkmark}$

Tool Time, Continued

② OLT : <u>Order Limit Theorem.</u>

Suppose $a_n \le b_n$ for all $n \in \mathbb{N}$. If $\lim a_n$ & $\lim b_n$ exist, then $\lim a_n \le \lim b_n$.

<u>Comments:</u>

① For example, if $\lim a_n$ exists & $a_n \le c$ $\forall n$, where $c \in \mathbb{R}$, then $\lim a_n \le c$.

(This is the special case of the theorem where $b_n = c$ $\forall n$.

(2) The theorem can be proved by first proving that if $a_n \geq 0$ $\forall n$ and $\lim a_n$ exists, then

$$\lim a_n \geq 0.$$

reason: let $A_n = b_n - a_n$.
By ALT, $\lim A_n = \lim b_n - \lim a_n$.

$b_n \geq a_n \Leftrightarrow A_n \geq 0$. If we prove $\lim A_n \geq 0$, then

$$\lim b_n - \lim a_n \geq 0$$
$$\Rightarrow \lim b_n \geq \lim a_n.$$

(3) This is false:
If $a_n < b_n$ $\forall n \in \mathbb{N}$ and $\lim a_n$ & $\lim b_n$ exist, then
$\lim a_n < \lim b_n$. Bad!

Counterexample: Let $a_n = 0$
$b_n = \frac{1}{n}$.

$\lim a_n = 0$

$\lim b_n = 0$ but $a_n < b_n$.

This *is* a valid example of the OLT, because $a_n \leq b_n$ and $\lim a_n$ & $\lim b_n$ exist, and $\lim a_n \leq \lim b_n$.

Exercise 2.3.6. Consider the sequence given by $b_n = n - \sqrt{n^2 + 2n}$. Taking $(1/n) \to 0$ as given, and using both the Algebraic Limit Theorem and the result in Exercise 2.3.1, show $\lim b_n$ exists and find the value of the limit.

Scratch

$b_n = n - \sqrt{n^2 + 2n}$

Given $\frac{1}{n} \to 0$

$$b_n = \frac{(n - \sqrt{n^2+2n})(n + \sqrt{n^2+2n})}{n + \sqrt{n^2+2n}}$$

$$b_n = \frac{n^2 - (n^2 + 2n)}{n + \sqrt{n^2+2n}} = \frac{-2n}{n + \sqrt{n^2+2n}}$$

$$b_n = \frac{-2n}{n(1) + n\sqrt{1 + 2/n}}$$

$$\Rightarrow b_n = \frac{-2}{1+\sqrt{1+2/n}}$$

Proof: Observe that

$$b_n = n - \sqrt{n^2+2n} = \frac{(n-\sqrt{n^2+2n})(n+\sqrt{n^2+2n})}{(n+\sqrt{n^2+2n})}$$

$$= \frac{n^2-(n^2+2n)}{n+\sqrt{n^2+2n}} = \frac{-2n}{n(1+\sqrt{1+2/n})}$$

$$\Rightarrow b_n = \frac{-2}{1+\sqrt{1+2/n}}.$$

Since $\lim \frac{1}{n} = 0$, $\lim 2 = 2$,

$\lim \frac{2}{n} = 2 \cdot 0 = 0$. by ALT

Since $\lim \frac{2}{n} = 0$ and $\lim 1 = 1$,

$\lim 1 + \frac{2}{n} = 1 + 0 = 1$. by ALT.

Since $\lim (1+\frac{2}{n}) = 1$ and $1+\frac{2}{n} \geq 0$,

$\lim \sqrt{1+\frac{2}{n}} = \sqrt{1} = 1$. by ALT

Since $\lim 1 = 1$ & $\lim \sqrt{1+\frac{2}{n}} = 1$,

$\lim 1 + \sqrt{1+\frac{2}{n}} = 1 + 1 = 2$. by ALT

Since $1+\sqrt{1+\frac{2}{n}} \neq 0$ and $\lim 1+\sqrt{1+\frac{2}{n}} = 2 \neq 0$

$$\lim \frac{1}{1+\sqrt{1+\frac{2}{n}}} = \frac{1}{2} = \frac{1}{2}.$$ by ALT

Since $\lim(-2) = -2$ &

$$\lim \frac{1}{1+\sqrt{1+\frac{2}{n}}} = \frac{1}{2},$$

$$\lim b_n = \lim \frac{-2}{1+\sqrt{1+\frac{2}{n}}} = (-2)\left(\frac{1}{2}\right)$$

by ALT.

$$= \boxed{-1}. \blacksquare$$

Exercise 2.3.7. Give an example of each of the following, or state that such a request is impossible by referencing the proper theorem(s):

(a) sequences (x_n) and (y_n), which both diverge, but whose sum $(x_n + y_n)$ converges;

(b) sequences (x_n) and (y_n), where (x_n) converges, (y_n) diverges, and (x_n+y_n) converges;

(a) e.g. $x_n = n$, $y_n = -n$ } Sp. $\lim x_n = L$, $\exists \varepsilon > 0$ s.t. $\forall N \in \mathbb{N}$, $\exists n \geq N$ s.t. $|x_n - L| \geq \varepsilon$.

$\varepsilon = 1$, $n = \lceil L \rceil \pm N$

Proof: Let $x_n = n$, $y_n = -n$ $\forall n \in \mathbb{N}$.

(x_n) diverges, because if $\lim x_n = L$,

let $\varepsilon = 1$. Then $\forall N \in \mathbb{N}$, let $n = \lceil |L| \rceil + N$

$\geq N$. Then $|x_n - L| = |\lceil |L| \rceil + N - L|$

$$\geq ||L| + N - L| \geq N \geq 1.$$

Thus (x_n) does not converge.

Similarly (y_n) diverges.

But $\lim(x_n + y_n) = \lim(n + -n)$

$$= \lim 0 = 0. \quad \blacksquare$$

(b)